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# On the zeroes of Artin $L$ -series of irreducible characters of the symmetric group $S_n$

Gerhard O. Michler

## 1 Introduction

Let  $E/F$  be a finite normal extension of algebraic number fields with Galois group  $\text{Gal}(E/F) = G$ . E. Artin [1] constructed to each virtual complex character  $\eta$  of  $G$  an  $L$ -series  $L(s, \eta, E/F)$  which is meromorphic in the whole complex plane  $\mathbb{C}$  as was proved by R. Brauer [3] by means of his famous induction theorem and fundamental classical results of E. Artin and E. Hecke. So far, no counterexample has been found to E. Artin's conjecture [1] which asserts that for each complex character  $\chi$  of  $G$  its  $L$ -series  $L(s, \chi, E/F)$  is holomorphic in  $\mathbb{C} - \{1\}$ . He showed that the Dedekind zeta function  $\zeta_F(s) = L(s, 1_G, E/F)$  of the field  $F$  has a pole of order 1 at  $s = 1$ , where  $1_G$  denotes the trivial character of  $G$ .

According to E. Hasse [6], p.163, it is also conjectured that in the vertical strip  $0 < \text{Re}(s) < 1$  of the complex plane  $\mathbb{C}$  the  $L$ -series  $L(s, \chi, E/F)$  of all characters  $\chi$  of  $G$  have all their zeroes on the line  $\text{Re}(s) = \frac{1}{2}$ . Riemann's conjecture for the classical zeta function  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$  is a special case of this conjecture, because  $\zeta(s)$  is the Dedekind zeta function for  $E = F = \mathbb{Q}$ , the field of rational numbers.

In this note we consider finite normal extensions  $E/F$  of algebraic number fields with Galois group  $\text{Gal}(E/F) = S_n$ , the symmetric group of degree  $n$ . Let  $k = [\frac{n}{2}]$ . In Theorem 5.3 it is shown that the truth of Artin's conjecture would imply that all the zeroes of the  $L$ -series  $L(s, \chi, E/F)$  of all irreducible characters  $\chi$  of  $S_n$  are contained in the union of the set of zeroes of the Dedekind zeta function  $\zeta_{\Omega}(s)$  of the proper intermediate field  $\Omega = E^{V_k}$  corre-

sponding to the wreath product  $V_k = C_2 \wr S_k$  of the cyclic group  $C_2$  of order 2 with the symmetric group  $S_k$ , and the union of sets of zeroes of the  $L$ -series of the sign characters  $\sigma_{n-2t}$  of the symmetric groups  $S_{n-2t}$  for  $0 \leq t \leq k-1$ . Furthermore, if also Riemann's conjecture holds for these  $k+1$   $L$ -series  $\zeta_\Omega(s)$  and  $L(s, \sigma_{n-2t})$ , then the zeroes of the  $L$ -series  $L(s, \chi, E/F)$  of all irreducible characters  $\chi$  of the symmetric group  $S_n$  with  $0 < \operatorname{Re}(s) < 1$  lie on the vertical line  $\operatorname{Re}(s) = \frac{1}{2}$ , see Corollary 5.4. This shows that the Dedekind zeta function  $\zeta_\Omega(s)$  of finite normal extensions  $E/\Omega$  of algebraic number fields with Galois group  $\operatorname{Gal}(E/\Omega) = C_2 \wr S_k$ ,  $k = 1, 2, \dots$ , and the  $L$ -series  $L(s, \sigma_n, E/F)$  of the sign characters  $\sigma_n$  of  $\operatorname{Gal}(E/F) = S_n$ ,  $n = 1, 2, \dots$ , are the critical cases for the so called Riemann's conjecture on the zeroes of the Artin  $L$ -series of the irreducible characters  $\chi$  of  $S_n$ .

In Theorem 5.2 the truth of Artin's conjecture is not assumed. It asserts that for each point  $s_0$  of  $\mathbb{C} - \{1\}$  there are at least  $k+1$  irreducible characters  $\chi_\nu$  of  $S_n$  whose  $L$ -series  $L(s, \chi, E/F)$  are holomorphic at  $s_0$ . The partitions  $\nu \vdash n$  parametrizing these  $k+1$  irreducible characters have different numbers of odd parts.

As in Foote-Murty [4] and Foote-Wales [5] Heilbronn's virtual character  $\Theta_G$  of  $G = \operatorname{Gal}(E/F)$  [7] is used essentially in the proofs of these results. Its main properties are described in section 3. Another important tool is the explicit model for the complex characters of the symmetric group  $S_n$  given by Inglis, Richardson and Saxl [8]. This is a set  $\{\pi_{t,n-2t} \mid 0 \leq t \leq k\}$  of monomial representations  $\pi_{t,n-2t}$  of  $S_n$  which together contain each irreducible representation  $\chi \in \operatorname{Irr}_{\mathbb{C}}(S_n)$  of  $S_n$  exactly once. The main result of [8] is explained in section 4. The basic definitions and properties of the Artin  $L$ -series  $L(s, \eta, E/F)$  are stated in section 2.

Concerning notation and terminology of the representation theory of finite groups we refer to the books by Nagao and Tsushima [11], and James and Kerber [9]. The standard reference for the results in algebraic number theory is S. Lang's book [10].

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## 2 Artin L-functions

In this section the basic definitions and notations from representation theory and number theory are given.

Let  $G$  be a finite group, then  $k(G)$  denotes the number of conjugacy classes of  $G$ . The set of all inequivalent irreducible characters of  $G$  is denoted by  $\text{Irr}G$ . In particular, we write  $\text{Irr}G = \{\chi_i \mid 1 \leq i \leq k(G)\}$ . The set  $\text{char}(G) = \{\sum_{i=1}^{k(G)} n_i \chi_i \mid n_i \geq 0, n_i \in \mathbb{Z}\}$  is the set of all ordinary characters.  $\text{vchar}(G) = \{\sum_{i=1}^{k(G)} n_i \chi_i \mid n_i \in \mathbb{Z}\}$  is the ring of all virtual characters.

**Definition.** Let  $p$  be a prime number. A subgroup  $H$  is called  $p$ -elementary, if  $H = P \times C$ , where  $P$  is a  $p$ -subgroup and  $C$  is a cyclic  $p'$ -subgroup of  $G$ .  $\mathcal{E}_p = \{H \mid H \text{ } p\text{-elementary subgroup of } G\}$ .  $\mathcal{E} = \bigcup_p \mathcal{E}_p$  is the set of all elementary subgroups of  $G$ .

**Brauer's Induction Theorem.** For each  $\eta \in \text{vchar}(G)$  there are elementary subgroups  $H_i \in \mathcal{C}$  and linear characters  $\lambda_{ij} \in \text{Irr}H_i$ ,  $1 \leq j \leq h_i$  such that

$$\eta = \sum_{i=1}^s \sum_{j=1}^{h_i} a_{ij} \lambda_{ij}^G$$

for some integers  $a_{ij} \in \mathbb{Z}$ .

A short proof of this fundamental result in the representation theory of finite groups is given in [11], p.207.

Let  $E/F$  be a finite normal extension of number fields  $E, F$  with Galois group  $\text{Gal}(E/F) = G$ . Let  $\mathcal{O}_F$  and  $\mathcal{O}_E$  be the ring of algebraic integers in  $F$  and  $E$ , respectively.

**Definition.** Let  $\eta \in \text{char}(G)$ . Let  $\mathcal{P}$  be the set of prime ideals  $p$  of  $\mathcal{O}_F$ . Then each  $p \in \mathcal{P}$  splits into a product

$$p = (P_1 \dots P_r)^e$$

of prime ideals  $P \in \{P_i \mid 1 \leq i \leq r\}$  of  $\mathcal{O}_E$ . If  $f$  is the degree of the residue class field extension then  $|E : F| = efr$  by [10], p.26.

For any  $P$  the norm  $NP = (Np)^f$ , where  $Np = |\mathcal{O}_F/p|$ . Let  $G_P$  be the inertial group of  $P$  in  $G$ . Let  $T_P = \{\tau \in G_P \mid \tau \text{ induces the identity automorphism}$

of the residue class field extension}. Then the Frobenius automorphism  $\sigma = \sigma(P, E/F) = \sigma(P, E/F) \in G_P$  is defined by

$$\sigma\alpha = \alpha^{N_P} \text{ mod } P, \quad \alpha \in \mathcal{O}_E.$$

$\sigma$  is determined only up to multiplication with some  $\tau \in T_P$ . For each  $m \geq 1$  and  $\sigma = (P, E/F)$  let  $\eta(\sigma^m T_P) = \sum_{t \in T_P} \eta(\sigma^m \tau)$ , and

$$\eta(p^m) = \frac{1}{e} \eta(\sigma^m T_P), \quad p \in \mathcal{P}.$$

Then the L-series  $L(s, \eta, E/F)$  is defined by

$$\log L(s, \eta, E/F) = \sum_{p \in \mathcal{P}} \sum_{m \geq 1} \frac{\eta(p^m)}{m(N_P)^{sm}}$$

$L(s, \eta, E/F)$  is holomorphic in the half plane  $\text{Re}(s) > 1$ . It has a continuation to the entire plane  $\mathbb{C}$ .

In [1] and [2] E. Artin proved or stated the following fundamental results. For precise references for its complete proof see also Foote-Wales [5], p.227.

**Lemma 2.1.** The L-series have the following properties:

1.  $L(s, \eta_1 \oplus \eta_2, E/F) = \Pi_{i=1}^2 L(s, \eta_i, E/F)$ , for all  $\eta_i \in \text{char} G$ .
2. If  $H \leq G$ ,  $\sigma \in \text{char}(H)$ , then  $L(s, \sigma^G, E/F) = L(s, \sigma, E/E^H)$ , where  $\sigma^G$  denotes the induced character of  $G$ .
3. For  $\psi \in \text{char} G$  let  $H = \ker \psi$ , and  $\psi'$  the character of  $G/H$  induced by  $\psi$ , then  $L(s, \psi, E/F) = L(s, \psi', E^H/F)$ .
4. (Hecke) If  $\chi$  is a non-principal linear character of  $G$ , then  $L(s, \chi, E/F)$  is holomorphic in the entire complex plane  $\mathbb{C}$ .
5. The Dedekind zeta function  $\zeta_F(s) = L(s, 1_G, E/F)$  has a simple pole at  $s = 1$ ,  $\zeta_F(1) \neq 0$ , and  $\zeta_F(s)$  is holomorphic everywhere except for  $s = 1$ .

6. Let  $\chi \in \text{Irr}G$  and  $\bar{\chi}$  be its complex conjugate. Artin multiplies  $L(s, \chi, E/F)$  and  $L(s, \bar{\chi}, E/F)$  with appropriate powers of the  $\Gamma$ -function  $\Gamma(s)$  and obtains meromorphic functions  $\xi(s, \chi, E/F)$  and  $\xi(s, \bar{\chi}, E/F)$  satisfying a functional equation

$$\xi(1-s, \chi, E/F) = W(\chi)\xi(s, \bar{\chi}, E/F),$$

where  $\xi(s, \chi, E/F)$  and  $L(s, \chi, E/F)$  have the same zeroes in

$$0 < \text{Re}(s) < 1.$$

**Remark 2.2.** If the Galois group  $\text{Gal}(E/F) = S_n$ , the symmetric group of degree  $n$ , then assertion 6 of Lemma 2.1 implies that in the vertical strip  $0 < \text{Re}(s) < 1$  the zeroes of all  $L$ -series  $L(s, \chi, E/F)$  of all irreducible characters  $\chi$  of  $S_n$  lie symmetric with respect to the vertical line  $\text{Re}(s) = \frac{1}{2}$ , because by Theorem 2.1.12 of James and Kerber [9], p.37 the rational field  $\mathbf{Q}$  is a splitting field for  $S_n$ , which implies  $\chi(g) = \bar{\chi}(g)$  for all  $g \in S_n$ .

In [3] R. Brauer proved the following fundamental results on the Artin  $L$ -functions  $L(s, \chi, E/F)$  by means of Lemma 2.1 and his induction theorem.

**Theorem 2.3.** The Artin  $L$ -series  $L(s, \chi, E/F)$ ,  $\chi \in \text{char}(G)$ , are all meromorphic in the complex plane  $\mathbf{C}$ .

**Artin's conjecture:** Let  $\eta \in \text{char}(G)$ . If the inner product  $\langle 1_G, \eta \rangle = 0$ , then  $L(s, \eta, E/F)$  has an analytic continuation for all  $s \in \mathbf{C}$ .

### 3 Heilbronn's virtual character

In [4] Foote and Murty showed that the set of zeroes and poles of the Artin  $L$ -functions  $L(s, \chi, E/F)$ ,  $\chi \in \text{char}(G)$ , are contained in the set of zeroes of the Dedekind zeta function  $\zeta_E(s)$  of the extension field  $E$ . In the proof of this result they apply some subsidiary results on a virtual character, originally introduced by H. Heilbronn [7]. Its definition and properties are restated in this section.

Let  $s_0 \in \mathbb{C} - \{1\}$  be fixed. For each  $\psi \in \text{Irr}G$  let  $n_\psi(s_0) = n_\psi = \text{ord}_{s=s_0} L(s, \psi, E/F)$  be the order of zero or pole of the meromorphic function  $L(s, \psi, E/F)$  at the point  $s_0$ . Heilbronn's virtual character is defined in [7], p.871, by

$$\Theta_G = \sum_{\psi \in \text{Irr}(G)} n_\psi \psi,$$

The following subsidiary results are due to Heilbronn [7], Foote-Murty [4] and Foote-Wales [5].

**Lemma 3.1.** a)  $\Theta_G$  is a virtual character of  $G$ .  
b) For each  $\psi \in \text{char}(G)$

$$\text{ord}_{s=s_0} L(s, \psi, E/F) = \langle \Theta_G, \psi \rangle.$$

Assertion a) is proved in [4], p.116, and b) is shown in [5], p.228.

Lemma 1 of Foote-Wales [5], p.230, is restated as

**Lemma 3.2.** For each subgroup  $H$  of  $G$  the restriction  $\Theta_{G|H} = \Theta_H$ .

**Lemma 3.3.** If  $\zeta_E(s)$  is the Dedekind zeta function of  $E$ , then

$$\Theta_G(1) = \text{ord}_{s=s_0} \zeta_E(s) \geq 0.$$

This result is proved in [5], p.228.

## 4 The model of the symmetric group

In [8] Inglis, Richardson and Saxl constructed an explicit model for the irreducible representations of the symmetric group  $S_n$ . It consists of a finite set of monomial representations defined over the integers  $\mathbf{Z}$ .

It is well known that the field  $\mathbf{Q}$  of rational numbers is a splitting field for any symmetric group  $S_n$ .

Throughout this section the integer  $n$  is fixed. Let  $A_n$  be the alternating subgroup of  $S_n$ . The irreducible representations of  $S_n$  are parametrized by the partitions  $\lambda \vdash n$  of  $n$ . The set of all partitions  $\lambda$  of  $n$  is denoted by  $\mathcal{P}(n)$ , and its cardinality  $|\mathcal{P}(n)|$  by  $p(n)$ . If  $\chi_\lambda$  is the character corresponding to  $\lambda \vdash n$ , then  $d_\lambda = \chi_\lambda(1)$  is the degree of  $\chi_\lambda$ .

The construction of the monomial representations of  $S_n$  given in [8] requires the following subgroups of  $S_n$  and linear (one dimensional) representations. Let  $t$  be any integer with  $0 \leq t \leq \lfloor \frac{n}{2} \rfloor$ . Let  $V_t = C_2 \wr S_t$  be the wreath product of the cyclic group  $C_2$  of order 2 with the symmetric group  $S_t$  of degree  $t$ . Let  $U_t = V_t \times S_{n-2t}$ , and  $W_t = V_t \times A_{n-2t}$ . Let  $\sigma_{n-2t}$  be the sign character of  $S_{n-2t}$ , and  $1_t$  the trivial representation of  $V_t$ . Then  $\mu_t = 1_t \otimes \sigma_{n-2t}$  is a linear representation of  $U_t$ . Therefore, the induced representation  $\pi_{t,n-2t} = (\mu_t)^{S_n}$  is a monomial representation of  $S_n$ . Furthermore,  $\pi_{t,n-2t}$  has degree  $m_t = \dim_F \pi_{t,n-2t} = \frac{n!}{2^t t! (n-2t)!}$ . This notation is kept throughout this section. The Corollary of Inglis, Richardson and Saxl [8] is restated as

- Proposition 4.1.**    a) The representation  $\sum_{0 \leq t \leq \lfloor \frac{n}{2} \rfloor} \pi_{t,n-2t}$  of  $S_n$  is the direct sum of all irreducible representations of  $S_n$ , each appearing with multiplicity one.
- b) The irreducible character  $\chi_\lambda$  of  $S_n$  corresponding to the partition  $\lambda$  of  $n$  is a constituent of  $\pi_{t,n-2t}$  if and only if  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$  has precisely  $n - 2t$  odd parts.

**Remark.**    In the special cases  $n = 2t$  and  $n = 2t + 1$ , the monomial representations  $\pi_{t,0}$  and  $\pi_{t,1}$  are in fact transitive permutation representations of  $S_n$ .



## 5 Artin's conjecture and the zeroes of the L-series of the irreducible characters of $S_n$

Throughout this section  $E/F$  denotes a finite normal extension of algebraic number fields  $E$  and  $F$  with Galois group  $Gal(E/F) = S_n$ , the symmetric group of degree  $n$ .

The L-series  $L(s, \pi_{t,n-2t}, E/F)$  of the monomial model characters  $\pi_{t,n-2t}$  of  $S_n$  are described by

**Lemma 5.1.** Let  $t$  be any integer with  $0 \leq t \leq [\frac{n}{2}]$ . Let  $V_t = C_2 \wr S_t$ ,  $U_t = V_t \times S_{n-2t}$ , and  $W_t = V_t \times A_{n-2t}$ . Let  $\sigma_{n-2t}$  be the sign character of  $S_{n-2t}$ ,  $1_t$  be the trivial character of  $V_t$ , and  $\mu_t = 1_t \otimes \sigma_{n-2t}$ . Then the following assertions hold:

a) If  $0 \leq n - 2t \leq 1$ , then  $L(s, \pi_{t,n-2t}, E/F) = L(s, (1_t)^{S_n}, E/F) = L(s, 1_t, E/E^{U_t}) = \zeta_\Omega(s)$ , where  $\zeta_\Omega(s)$  denotes the Dedekind zeta function of the intermediate field  $\Omega = E^{U_t}$  corresponding to the subgroup  $U_t = V_t = C_2 \wr S_t$  of  $S_n$ .

b) If  $n - 2t > 1$ , then  $L(s, \pi_{t,n-2t}, E/F) = L(s, (\mu_t)^{S_n}, E/F) = L(s, \mu_t, E/E^{U_t}) = L(s, \sigma_{n-2t}, E^{V_t}/E^{U_t}) = L(s, \sigma'_{n-2t}, E^{W_t}/E^{U_t})$ , where  $Gal(E^{V_t}/E^{U_t}) \cong S_{n-2t}$  and  $|Gal(E^{W_t}/E^{U_t})| = 2$ .

**Proof.** a) follows immediately from assertions (2) and (5) of Lemma 2.1.

b) Certainly  $L(s, (\mu_t)^{S_n}, E/F) = L(s, \mu_t, E/E^{U_t})$  by (2) of Lemma 2.1. The linear character  $\mu_t = 1_t \times \sigma_{n-2t}$  of  $U_t = V_t \times S_{n-2t}$  has  $V_t$  in its kernel, and it induces the sign character  $\sigma_{n-2t}$  in the factor group  $U_t/V_t \cong S_{n-2t}$ . Therefore, assertion (3) of Lemma 2.1 implies that

$$L(s, \mu_t, E/E^{U_t}) = L(s, \sigma_{n-2t}, E^{V_t}/E^{U_t}).$$

Furthermore,  $Gal(E^{V_t}/E^{U_t}) \cong S_{n-2t}$  by the main theorem of Galois theory. As  $A_{n-2t} = \ker(\sigma_{n-2t}) \triangleleft S_{n-2t}$ , another application of Lemma 2.1 (3) yields that  $L(s, \sigma_{n-2t}, E^{V_t}/E^{U_t}) = L(s, \sigma'_{n-2t}, E^{W_t}/E^{U_t})$ , where  $\sigma_{n-2t}$  denotes the non-trivial character of the cyclic group  $U_t/W_t \cong S_{n-2t}/A_{n-2t}$  of order 2. This completes the proof.

**Theorem 5.2.** Let  $t$  be any integer with  $0 \leq t \leq [\frac{n}{2}]$ , and  $s_0$  any point of the Gaussian plane  $\mathbb{C} - \{1\}$ . Then for at least one irreducible character  $\chi_\nu$

such that  $\nu \vdash n$  has  $n-2t$  odd parts, the  $L$ -series  $L(s, \chi, E/F)$  is holomorphic at  $s_0$ .

Proof. Let  $k(t)$  be the number of irreducible characters  $\chi_{\nu_i}$  of  $S_n$  corresponding to the partitions  $\nu_i \vdash n$  of  $n$  with precisely  $n-2t$  odd parts. Then

$$\pi_{t,n-2t} = (\mu_t)^{S_n} = \sum_{i=1}^{k(t)} \chi_{\nu_i}$$

by Proposition 4.1, where  $\mu_t = 1_t \otimes \sigma_{n-2t}$  denotes the linear character of  $U_t = V_t \times S_{n-2t}$  described in the previous section. By Brauer's theorem 2.3, all  $L$ -series  $L(s, \chi_{\nu_i}, E/F)$  are meromorphic at  $s_0$ . Let  $n_i$  be the order of a pole or a zero of  $L(s, \chi_{\nu_i}, E/F)$  at  $s_0$ , and let  $\Theta = \sum_{\psi \in Irr(S_n)} n_\psi \psi$  be Heilbronn's virtual character of  $S_n$  with respect to  $s_0$ . As  $L(s, \pi_{t,n-2t}, E/F)$  is holomorphic at  $s_0$  by assertions (2) and (4) or (5) of Lemma 2.1, it follows from Lemma 3.1 b) that

$$0 \leq ord_{s=s_0} L(s, \pi_{t,n-2t}, E/F) = \langle \Theta, \pi_{t,n-2t} \rangle = \sum_{i=1}^{k(t)} n_i.$$

Therefore, at least one  $n_i \geq 0$  for some  $1 \leq i \leq k(t)$ . Thus  $L(s, \chi_{\nu_i}, E/F)$  is holomorphic at  $s_0$ .

**Theorem 5.3.** Let  $E/F$  be a finite normal extension of algebraic number fields with Galois group  $Gal(E/F) = S_n$ . For each  $0 \leq t \leq [\frac{n}{2}] = k$  let  $V_t = C_2 \wr S_t$ ,  $U_t = V_t \times S_{n-2t}$  and  $\sigma_{n-2t}$  be the sign character of the symmetric group  $S_{n-2t}$ . Let  $\zeta_\Omega(s)$  be the Dedekind zeta function of the intermediate field  $\Omega = E^{V_k}$ .

If the  $L$ -series  $L(s, \chi, E/F)$  of all the irreducible characters  $\chi$  of  $S_n$  are holomorphic in  $\mathbb{C} - \{1\}$  then the zeroes of all  $L$ -series  $L(s, \chi, E/F)$  are contained in the set of zeroes of the Dedekind zeta function  $\zeta_\Omega(s)$  and of the  $k$  Artin  $L$ -series  $L(s, \sigma_{n-2t}, E^{V_t}/E^{U_t})$  of the sign characters  $\sigma_{n-2t}$  of the Galois groups  $Gal(E^{V_t}/E^{U_t}) \cong S_{n-2t}$ ,  $0 \leq t \leq k-1$ .

Proof. Let  $s_0$  be a point in  $\mathbb{C} - \{1\}$  such that  $\zeta_\Omega(s_0) \neq 0$  and  $L(s_0, \sigma_{n-2t}, E^{V_t}/E^{U_t}) \neq 0$  for all  $0 \leq t \leq k-1$ . Let  $\chi$  be any irreducible character of  $S_n$ . Then there is a uniquely determined partition  $\nu \vdash n$  corresponding to  $\chi = \chi_\nu$ . Suppose that  $\nu$  has  $n-2t$  odd parts. Then by

Proposition 4.1  $\chi_\nu$  occurs in the monomial model character  $\pi_{t,n-2t}$  of  $S_n$  with multiplicity 1, and  $\langle \chi_\nu, \pi_{s,n-2s} \rangle = 0$  for all  $0 \leq s \leq k$  and  $s \neq t$ .

Let  $k(t)$  be the number of irreducible characters  $\chi_{\nu_i}$  of  $S_n$  corresponding to the partitions  $\nu_i \vdash n$  of  $n$  with precisely  $n-2t$  parts. We may assume that  $\nu = \nu_1$ . Let  $n_i$  be the order of a zero of the holomorphic function  $L(s, \chi_{\nu_i}, E/F)$  at  $s_0$ , and let  $\Theta = \sum_{\psi \in \text{Irr}(S_n)} n_\psi \psi$  be Heilbronn's virtual character of  $S_n$  with respect to  $s_0$ . Then  $n_i \geq 0$  for  $i = 1, 2, \dots, k(t)$ , and by Lemma 3.1 b)

$$\text{ord}_{s=s_0} L(s, \pi_{t,n-2t}, E/F) = \langle \Theta, \pi_{t,n-2t} \rangle = \sum_{i=1}^{k(t)} n_i.$$

Now Lemma 5.1 asserts that

$$\text{ord}_{s=s_0} L(s, \pi_{t,n-2t}, E/F) = \text{ord}_{s=s_0} \zeta_\Omega(s) = 0 \text{ for } t = k, \text{ and}$$

$$\text{ord}_{s=s_0} L(s, \pi_{t,n-2t}, E/F) = \text{ord}_{s=s_0} L(s, \sigma_{n-2t}, E^{V_t}/E^{U_t}) = 0,$$

because  $\zeta_\Omega(s_0) \neq 0$ ,  $L(s_0, \sigma_{n-2t}, E^{V_t}/E^{U_t}) \neq 0$  and both functions are holomorphic at  $s_0$ . Hence, all  $n_i = 0$  for  $1 \leq i \leq k(t)$ . In particular,  $L(s_0, \chi, E/F) \neq 0$ , completing the proof.

**Corollary 5.4.** Let  $E/F$  be a finite normal extension with Galois group  $\text{Gal}(E/F) = S_n$  such that the  $L$ -series  $L(s\chi, E/F)$  of all irreducible characters  $\chi$  of the symmetric group  $S_n$  are holomorphic in  $\mathbb{C} - \{1\}$ . Let  $k = [\frac{n}{2}]$ . If Riemann's conjecture holds for

a) all  $L$ -series  $L(s, \sigma_{n-2t}, E_t/F_t)$  of the sign characters  $\sigma_{n-2t}$  of the symmetric groups  $S_{n-2t}$  and all finite extensions  $E_t/F_t$  with Galois groups  $\text{Gal}(E_t/F_t) = S_{n-2t}$  for  $0 \leq t \leq k-1$ , and

b) the Dedekind zeta function  $\zeta_\Omega(s)$  of all finite normal extensions  $E_k/F_k$  with Galois group  $\text{Gal}(E_k/F_k) = C_2 \wr S_k$ ,

then the zeroes of the  $L$ -series  $L(s, \chi, E/F)$  of all irreducible characters  $\chi$  of  $S_n$  with  $0 < \text{Re}(s) < 1$  lie on the vertical line  $\text{Re}(s) = \frac{1}{2}$ .

Proof follows immediately from Theorem 5.3

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